PhD Analysis Examination
May 2022

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Attempt SIX problems.

1. State Tonelli's theorem, and give an example to show that the σ-finiteness hypothesis is necessary.

2. Let μ be a positive measure and let (E_n) be a sequence of measurable sets with $\mu(E_n) < \infty$ for all n. Prove that if $\mathbf{1}_{E_n} \to f$ in $L^1(\mu)$, then there exists a measurable set E so that $f = \mathbf{1}_E$ a.e. and $\mu(E) = \lim \mu(E_n)$.

3. Let μ and ν be finite, positive measures defined on the same measurable space $(X, \mathcal{M})$, and suppose that $\nu \ll \mu$.

   a) Prove that if $L^1(\mu) \subset L^1(\nu)$, then the inclusion map $\iota : L^1(\mu) \hookrightarrow L^1(\nu)$ is necessarily bounded.

   b) Prove that if $L^1(\mu) \subset L^1(\nu)$ then $\frac{d\nu}{d\mu} \in L^\infty(\mu)$.

4. Let μ be a positive measure and suppose that $1 < p < r < q < \infty$. Prove that

$$L^r(\mu) \subset L^p(\mu) + L^q(\mu)$$

   (That is, every $f \in L^r(\mu)$ can be written as a sum $f = g + h$ where $g \in L^p(\mu)$ and $h \in L^q(\mu)$.)

5. Let $\mathcal{X}$ be a Banach space, and let $\mathcal{M}, \mathcal{N}$ be closed subspaces of $\mathcal{X}$. Suppose that every $x \in \mathcal{X}$ can be decomposed *uniquely* as

$$x = m + n$$

   with $m \in \mathcal{M}, n \in \mathcal{N}$. Prove that the assignment $x \to m$ defines a bounded linear operator from $\mathcal{X}$ to $\mathcal{M}$.

6. a) Prove that for each integer $n \geq 0$, there exists a function $f_n \in L^2[0,1]$ such that for every polynomial p of degree at most n,

$$p(1) = \int_0^1 p(x) f_n(x)\, dx.$$

   Is f_n unique?

b) Is there a function $f \in L^2[0,1]$ such that

$$p(1) = \int_0^1 p(x)f(x)\,dx$$

for *all* polynomials p? Prove or disprove.

7. a) State the Banach Isomorphism Theorem and the Closed Graph Theorem. b) Using the Banach Isomorphism Theorem (or otherwise), prove the Closed Graph Theorem.

8. (Short answer, you do not need to give a detailed proof, just a brief explanation.)

   a) Define the Fourier transform on $L^1(\mathbb{R})$ and sketch its extension to $L^2(\mathbb{R})$.

   b) Sketch a construction of a bounded linear functional λ on $L^\infty(\mathbb{R})$ which is *not* of the form $\lambda(f) = \int f(x)g(x)\,dx$ for any L^1 function g.

   c) Sketch a proof of the fact that there is no norm under which c_{00} is complete.