PhD Analysis Examination
May 2021

Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Attempt SIX problems.

1. Let $f \in L^1(\mathbb{R})$ and suppose that
$$\int_a^b f\,dm = 0 \quad \text{for all rational } a < b.$$
Does this imply that $f = 0$ a.e.? Prove, or give a counterexample.

2. Prove that if $E \subset \mathbb{R}$ has positive Lebesgue measure, then the set
$$E - E = \{x - y | x, y \in E\}$$
contains an open interval centered at 0.

3. Let μ and ν be nonzero, σ-finite measures defined on the same measurable space $(X, \mathcal{M})$, and suppose each is absolutely continuous with respect to the other. Prove that
$$\frac{d\nu}{d\mu} = \frac{1}{\left(\frac{d\mu}{d\nu}\right)} \quad \mu\text{-a.e.}$$

4. Let μ be a positive measure and suppose that $1 < p < r < q < \infty$. Prove that
$$L^p(\mu) \cap L^q(\mu) \subset L^r(\mu).$$

5. Let H be a Hilbert space and $M \subset H$ a norm-closed subspace. Prove that if (x_n) is a sequence from M and x_n converges *weakly* to $x \in H$, then in fact $x \in M$.

6. a) Prove that for each integer $n \geq 0$, there exists a finite Borel measure μ_n on $[0, 1]$ so that for every polynomial p of degree at most n,
$$p'(0) = \int_0^1 p(x)\,d\mu_n(x).$$
b) Prove that there is *no* finite Borel measure μ on $[0, 1]$ such that
$$p'(0) = \int_0^1 p(x)\,d\mu(x).$$
for *all* polynomials p.

7. a) State the Banach Isomorphism Theorem and the Closed Graph Theorem. b) Using the Banach Isomorphism Theorem (or otherwise), prove the Closed Graph Theorem.

8. (Short answer, you do not need to give a detailed proof, just a brief explanation.)

   a) Give an example, if possible, of a sequence $(f_n) \subset L^2(\mathbb{R})$ such that $f_n \to 0$ weakly, but not in measure.

   b) Give an example to show that the σ-finiteness hypothesis is necessary in Tonelli's theorem.

   c) If $(X, \mathcal{M})$ is a measurable space, μ is a signed measure on $(X, \mathcal{M})$, and μ omits the value $+\infty$, does μ attain a maximum value on $\mathcal{M}$?