ANALYSIS QUALIFYING EXAM MAY 2019

Do six problems.

(1) Let X, Y be topological spaces. Prove that if $f : X \to Y$ is continuous, then f is Borel measurable.

(2) Rigorously determine the following expressions:

(a)
$$\lim_{n\to\infty} \int_0^\infty \frac{\sin \frac{x}{n}}{(1+\frac{x}{n})^n} dx$$

(b)
$$\lim_{n\to\infty} \int_0^\infty \frac{n \sin \frac{x}{n}}{x(1+x^2)} dx$$

(c)
$$\lim_{n\to\infty} \int_0^\infty \frac{n}{n^2+x^2} dx$$

(3) Prove that if a sequence of functions converges in L^1, then it has a subsequence that converges pointwise almost everywhere.

(4) State and prove the Hahn decomposition theorem. [1]

(5) Let X be a Banach space. Let $Y, Z \subseteq X$ be closed subspaces such that $Y + Z = X$ and $Y \cap Z = \{0\}$. Prove there is $C > 0$ such that for all $y \in Y$ and $z \in Z$,

$$C(\|y\| + \|z\|) \le \|y + z\| \le \|y\| + \|z\|.$$

(6) Let X be a Banach space. Show that the natural map from X to X^{**} is injective.

(7) State and prove the Cauchy-Schwarz inequality over a complex Hilbert space.

(8) Prove or give a counterexample: If f, g are smooth functions on $\mathbb{R}$ such that $f, g \in L^1(\mathbb{R})$, then $f * g$ is a smooth function and $f * g \in L^1(\mathbb{R})$.

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[1] For the proof, you may want to consider a sequence of sets such that $\mu(E_n) \to \sup_E \mu(E)$ rapidly, under the assumption that such a supremum is bounded.