PhD Analysis Exam, August 2019

Do three from part A and three from part B. Answer each problem on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Part A

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(1) Let X, Y be topological spaces. Prove that every continuous function $f : X \to Y$ is Borel measurable.

(2) Short answer.

(a) Give an example, if possible, of a *closed* set $E \subset \mathbb{R}$ of positive Lebesgue measure that contains no interval.

(b) For $n \geq 2$, let $f_n = e^{inx} x^{-n}$. Find

$$\lim_{n\to\infty} \int_1^\infty f_n \, dx.$$

(c) Give an example, if possible, of strict inequality in Fatou's Lemma.

(3) Let $X = Y = [0,1]$, $\mathscr{M} = \mathscr{B}_{[0,1]}$ denote the Borel σ-algebra, and $\mathscr{N} = 2^{\mathbb{R}}$. Let μ Lebesgue measure on $\mathscr{M}$, and let ν counting measure on $\mathscr{N}$. Prove that $\Delta = \{(x,x) : x \in [0,1]\} \subset [0,1] \times [0,1]$ is an element of the product σ-algebra $\mathscr{M} \otimes \mathscr{N}$ and compute

$$\int_X \left(\int_Y \mathbf{1}_\Delta(x,y) \, d\nu(y) \right) d\mu(x), \quad \int_Y \left(\int_X \mathbf{1}_E(x,y) \, d\mu(x) \right) d\nu(y). \quad (1)$$

Explain the relation to Tonelli's Theorem on product integration.

(4) State the Hahn Decomposition Theorem. Prove, if ρ is a signed measure on the measurable space $(X, \mathscr{M})$, then there exist unique positive measures $\rho_\pm$ on $\mathscr{M}$ such that $\rho_+ \perp \rho_-$ and $\rho = \rho_+ - \rho_-$.

Assuming μ is a positive measure on $\mathscr{M}$ and $f \in L^1(\mu)$ is real-valued, identify $\rho_\pm$ for the signed measure

$$\rho(E) = \int_E f \, d\mu.$$

Part B

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5. State the Baire Category Theorem. Suppose X is a Banach space. Prove that, as a vector space, X does not have a countable basis. (Here countable means equinumerous with $\mathbb{N}$.)

6. Let (φ_n) be a sequence from $L^1(\mathbb{R})$ with the following properties:

   (i) $\varphi_n \geq 0$ for all n,

   (ii) $\int \varphi_n \, dx = 1$ for all n;

   (iii) for each $\delta > 0$, $\lim_{n\to\infty} \int_{|x|>\delta} \varphi_n \, dx = 0$.

   Show, if $f \in L^1(\mathbb{R})$, then $\varphi_n \star f$ converges to f in $L^1(\mathbb{R})$.

7. Let X be a Banach space, $Y \subset X$ a closed subspace. Say Y is *complemented* in X if there exists a *closed* subspace $Z \subset X$ such that $Y \cap Z = \{0\}$ and $Y + Z = X$.

   Prove that if Y is complemented in X, then there exists a bounded linear operator $T : X \to Y$ such that $T(y) = y$ for all $y \in Y$.

8. Let $S : \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$ denote the *shift operator* defined, for $f = (f(n))_{n=0}^{\infty} \in \ell^2(N)$, by

$$Sf(n) = \begin{cases} f(n-1) & n \geq 1; \\ 0 & n = 0. \end{cases}$$

   Show S is an isometry (and in particular is bounded) and find S^* and SS^*. Is S unitary?