PhD Analysis Exam, May 2017

Do six of eight

Work each problem on a separate sheet. Problems (2) and (5) are short answer. For the remaining problems, please provide complete proofs with all details.

(1) Prove that if X, Y are topological spaces and $f : X \to Y$ is continuous, then f is Borel measurable.

(2) (Do three of four.)
  (a) Give an example, if possible, of a closed set $E \subset \mathbb{R}$ of positive Lebesgue measure that contains no interval.
  (b) Give an example, if possible, of a sequence $f_n : [0, 1] \to \mathbb{R}$ of Lebesgue measurable functions that converges in measure, but not pointwise a.e.
  (c) Give an example, if possible, of a premeasure on a Boolean algebra $\mathscr{A}$ without a unique extension to a measure on the σ-algebra generated by $\mathscr{A}$.
  (d) Determine the limit of the sequence $(a_n)_{n=1}^{\infty}$ defined by

$$a_n = \int_1^{\infty} \frac{\cos^2(\pi t)}{t^n}\, dt.$$

(3) Suppose $(X, \mathscr{M}, \mu)$ is a σ-finite measure space and $\mathscr{N} \subset \mathscr{M}$ is a (sub) σ-algebra. Let $\nu = \mu|_{\mathscr{N}}$. Show, if $(X, \mathscr{N}, \nu)$ is a σ-finite measure space and $f \in L^1(\mu)$, then there exists a $g \in L^1(\nu)$ such that, for all $E \in \mathscr{N}$,

$$\int_E g\, d\nu = \int_E f\, d\mu.$$

Is the σ-finiteness hypothesis on ν necessary?

(4) Given $f \in L^1([0, 1])$, let $g(x) = \int_x^1 \frac{f(t)}{t}\, dt$. Show $g \in L^1([0, 1])$ and

$$\int_0^1 g = \int_0^1 f.$$

(5) (Do three of four)
  (a) Is there a norm on the vector space

$$c_{00} = \{f : \mathbb{N} \to \mathbb{C} : \text{ there exists an } N \text{ such that} f(n) = 0 \text{ for } n \geq N\}$$

  that makes c_{00} a Banach space?
  (b) Explain what is meant by the Fourier transform of a function $f \in L^2(\mathbb{R})$.
  (c) Is there bounded linear bijection between $C([0, 1])$ and $L^{\infty}([0, 1])$?
  (d) Give an example, if possible, of a sequence of unit vectors (f_n) from $L^2(\mathbb{R})$ that converges weakly to some $f \in L^2(\mathbb{R})$, but not in norm.

(6) Show, if $E \subset \mathbb{R}$ is a Lebesgue measurable set of positive measure, then $E - E = \{x - y : x, y \in E\}$ contains an open interval.

(7) Suppose $\mathscr{X}$ is a Banach space and $\mathscr{M}$ and $\mathscr{N}$ are closed subspaces. Show, if for each $x \in \mathscr{X}$ there exist unique $m \in \mathscr{M}$ and $n \in \mathscr{N}$ such that

$$x = m + n,$$

then the mapping $P : \mathscr{X} \to \mathscr{M}$ defined by $Px = m$ is bounded.

(8) Suppose $(X, \mathscr{M}, \mu)$ is a σ-finite measure space and $1 \leq q < p < \infty$. Show, $L^q \subset L^p$ if and only if there exists a $\delta > 0$ such that if $E \in \mathscr{M}$ then either $\mu(E) = 0$ or $\mu(E) \geq \delta$.