Ph. D. Algebra Exam **May 11, 2018**

Answer **seven** problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. (10 points) Determine the Galois group of the polynomial

$$f = x^4 - 2 \in \mathbb{Q}[x].$$

2. Let K be a field and $K(x)$ the field of rational functions over K. Let $\frac{f}{g} \in K(x)$, with $\frac{f}{g} \notin K$, where f and g are relatively prime elements of $K[x]$.

   (a) (5 points) Show that $\frac{f}{g}$ is transcendental over K.

   (b) (5 points) Prove that $K(x)$ is a finite extension of $K(\frac{f}{g})$.

3. Let $\mathcal{C}$ be a category and let Λ be a set. Suppose that for each $\lambda \in \Lambda$ we are given an object X_λ in $\mathcal{C}$.

   (a) (3 points) Give the definition of a *coproduct* of the collection of objects $\{X_\lambda\}_{\lambda\in\Lambda}$.

   (b) (2 points) Prove that any two coproducts of the $\{X_\lambda\}_{\lambda\in\Lambda}$ are isomorphic.

   (c) (5 points) Prove that in the category of Abelian groups a coproduct exists for any collection of objects indexed by a set Λ.

4. (10 points) Let F be the free group on two generators x and y. Prove that the commutator subgroup is not finitely generated.

5. Let A be a torsion abelian group, and let $\mathbb{Q}$ be the (additive) group of the rational numbers.

   (a) (5 points) Prove $A \otimes_{\mathbb{Z}} \mathbb{Q} = \{0\}$.

   (b) (5 points) Prove $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}$.

6. Let R be a ring and suppose that we have the following commutative diagram of R-modules and module homomorphisms with exact rows.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \xrightarrow[f]{} & B & \xrightarrow[g]{} & C & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\ 0 & \longrightarrow & A' & \xrightarrow[f']{} & B' & \xrightarrow[g']{} & C' & \longrightarrow & 0 \end{array}.$$

   (a) (5 points) Show that if α and γ are monomorphisms, then so is β.

   (b) (5 points) Show that if α and γ are epimorphisms, then so is β.

7. (10 points) Let R be a commutative ring with $1 \neq 0$. Prove that the set consisting of 0 and all zero divisors in R contains at least one prime ideal.

8. (10 points) State and prove Hilbert's Basis Theorem.

9. (10 points) Prove that an ideal of a Dedekind domain can be generated by two elements.

10. (a) (5 points) Prove Schur's Lemma: If R is a ring and M a simple R-module, then $D := \mathrm{Hom}_R(M, M)$ is a division ring.

    (b) (5 points) Give an example of R and M as above where D is not a field.

11. (10 points) Let V be a nonzero vector space over a field F. Prove that $R = \mathrm{Hom}_F(V, V)$ is a simple ring if and only if V is finite-dimensional over F.