Answer **seven** problems. (If you turn in more, the first seven will be graded.) Put your answers in numerical order and circle the numbers of the seven problems below your name. Within reason, you may quote theorems as long as you state them clearly.

Name: ______________________________

Problems to be graded: 1 2 3 4 5 6 7 8 9 10 11

Note. Below *ring* means associative ring with identity, and *module* means unital module.

1. (10 points) Let $q > 1$ be a power of a prime p. Prove that there exists some field F with $|F| = q$.

2. (10 points) Determine the Galois group of $x^4 - 5$ over $\mathbf{Q}$, over $\mathbf{Q}(\sqrt{5})$, and over $\mathbf{Q}(\sqrt{5}i)$. Justify your answers.

3. (10 points) Prove that if G is a finite multiplicative subgroup of a field F, then G is cyclic.

4. (10 points) Prove that if R is an integral domain and a local ring, and I invertible ideal of R, then I is principal.

5. (10 points) Let R be a commutative ring with identity, and let I and J be ideals of R. Prove that the R-modules $(R/I) \otimes_R (R/J)$ and $R/(I + J)$ are isomorphic.

6. (10 points) State and prove Hilbert's Basis Theorem.

7. (10 points) Prove the *Lying-Over Theorem*: Let S be an integral extension of an integral domain R, and let P be a prime ideal of R. Then, there exists a prime ideal Q of S, such that $Q \cap R = P$.

8. Let R be a ring.
   (a) (3 points) Define what it means for an R-module M to be *projective*.
   (b) (4 points) Prove that any free R-module is projective.
   (c) (3 points) Prove that there exists some ring R and some projective R-module P such that P is a projective module but not a free module.

9. (10 points) Let R be a commutative ring with identity. Let $\phi\colon R \to R$ be a ring homomorphism. Assume R is Noetherian and that ϕ is surjective. Prove that ϕ is an automorphism.

10. Let V be a finite dimensional non-zero vector space over a field F. Let $R = \mathrm{End}_F(V)$ be the ring of endomorphisms of V.
    (a) (7 points) Prove that R is a simple ring.
    (b) (3 points) Describe a minimal (non zero) left ideal of R.

11. Let $\mathcal{F}G$ be the category of all finite groups.
    (a) (5 points) Recall the definition of *free object* in an arbitrary concrete category $\mathcal{C}$.
    (b) (5 points) Prove from your definition that there exists a finite set S such that there is no free object in $\mathcal{F}G$ freely generated by S.