**Ph. D. Algebra Exam** **January 4th, 2019**

Answer seven problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Determine the Galois group of the polynomial

$$f(x) = x^4 - 14x^2 + 9 \in \mathbb{Q}[x].$$

2. (a) Give the definition of a *separable* field extension.
   (b) Show that a finite separable extension is contained in a finite Galois extension.
   (c) Prove the *Primitive Element Theorem*: If $E$ is a finite separable extension of $F$, then $E = F(\alpha)$ for some $\alpha \in E$.

3. A *pointed set* is a pair $(S, x)$ with $S$ a set and $x \in S$. A morphism of pointed sets $(S, x) \to (S', x')$ is a function $f : S \to S'$ such that $f(x) = x'$.
   (a) Show that pointed sets and their morphisms form a category.
   (b) Show that coproducts of arbitrarily indexed families of objects exist in this category.

4. Let $F$ be the free group on $n$ generators, $n \geq 1$.
   (a) Prove that the automorphism group of $F$ contains a subgroup isomorphic to the symmetric group $S_n$.
   (b) Give an example of an automorphsim of $F$ that does not belong to this subgroup.

5. Define the term *injective module* for a ring $R$ with 1. Prove *Baer's criterion*: A unitary left module $J$ is injective if and only if for every left ideal $L$ of $R$, any $R$-module homomorphism $L \to J$ can be extended to an $R$-module homomorphism $R \to J$.

6. Let $R$ be a ring and suppose that we have a short exact sequence

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

of left $R$-modules. Let $M$ be a right $R$-module. Show that the induced sequence

$$M \otimes_R A \xrightarrow{1 \otimes f} M \otimes_R B \xrightarrow{1 \otimes g} M \otimes_R C \to 0$$

is exact.

7. Let $R$ be a commutative ring with $1 \neq 0$ and let $S$ be a multiplicative set containing 1.

(a) State the universal mapping property of the localization $S^{-1}R$.

(b) Show that if $S = R \setminus P$, the complement of a prime ideal $P$, then $S^{-1}R$ has a unique maximal ideal.

8. Prove the following properties of $R = \mathbb{Z}[\sqrt{10}]$

   (a) $R$ is noetherian.

   (b) $R$ is an integrally closed integral domain.

   (c) Every nonzero prime ideal of $R$ is maximal.

9. Let $J$ be an ideal in a commutative ring $R$ with 1. Assume that $J$ is contained in every maximal ideal. Prove that if $A$ is an $R$-module that satisfies the ascending chain condition on submodules, then $\cap_{n=1}^{\infty} J^n A = 0$.

10. Prove the *Noether Normalization Lemma*: Let $R$ be an integral domain that is a finitely generated extension ring of a field $K$. Let $F$ be the field of quotients of $R$ and let $r$ be the transcendence degree of $F$ over $K$. Then there exist a subset of $r$ elements $t_1, \ldots, t_r$ which are algebraically independent over $K$ and such that $R$ is integral over $K[t_1, \ldots, t_r]$.

11. (a) Define the term *left primitive ring* and state *Jacobson's Density Theorem*.

    (b) Prove that if $R$ is a finite left primitive ring then $R \cong \mathrm{M}_n(F)$ for some finite field $F$ and some natural number $n$.