First-Year Second-Semester Exam
May 2018

Answer the following problems and show all your work. Support all statements to the best of your ability. Use a separate sheet of paper for each problem. Each is worth 10 points.

1. Let X be a completely regular space.
   (a) Define a compactification of X.
   (b) Define $\beta(X)$, a Stone-Čech compactification of X, in terms of its universal property.
   (c) Show that if $\beta(X)$ exists then it is unique up to homeomorphism.
2. Suppose that the continuous maps $f, g : X \to Y$ are homotopic and that the continuous maps $h, k : Y \to Z$ are homotopic. Show that $h \circ f$ is homotopic to $k \circ g$.
3. Show that the fundamental group of the circle is isomorphic the additive group of integers. That is, $\pi_1(S^1, 1) \cong \mathbb{Z}$.
4. State and prove the Brouwer fixed point theorem for the unit disk, D^2.
5. Prove that for $n \geq 2$, the fundamental group of the n-sphere, S^n, is 0.

Answer the following with complete definitions or statements or short proofs. Each is worth 5 points.

6. State the *Tietze Extension Theorem.*
7. Define a *Baire space.* State the *Baire category theorem.*
8. Suppose that $f, g : X \to \mathbb{R}^n$ are continuous functions. Show that f and g are homotopic.
9. Define the *retraction* of a topological space X onto a subspace A. If $j : A \hookrightarrow X$ is the inclusion of a retract, then show that the induced map on fundamental groups $j_\#$ is injective.
10. Show that S^1 is not homeomorphic to S^2.
11. Show that $\mathbb{R}^2$ is not homeomorphic to S^2.
12. Is the squaring map $s : S^1 \to S^1$, $s(z) = z^2$, nulhomotopic?
13. What is the fundamental group of the torus $T^2 = S^1 \times S^1$?
14. State the *Seifert-van Kampen Theorem.*
15. Describe a quotient of a polygon whose fundamental group is isomorphic to $\mathbb{Z}/3$.