First-Year First-Semester Exam
January 2018

Answer the following problems and show all your work. Support all statements to the best of your ability. Use a separate sheet of paper for each problem. Each is worth 10 points.

1. A function $f : \mathbb{N} \to \mathbb{Z}$ is called *eventually zero* if there is some N such that $f(n) = 0$ for all $n \geq N$. Show that the set of functions $f : \mathbb{N} \to \mathbb{Z}$ that is eventually zero is countable.
2. Let $f : X \to Y$ be continuous and onto. Prove that if X is compact then so is Y.
3. Let $f, g : X \to Y$ be continuous maps to a Hausdorff space. Prove that the set $\{x \mid f(x) = g(x)\}$ is a closed subset of X.
4. Prove that if X is a compact Hausdorff space, then X is a normal space.
5. State and prove the Contraction Mapping Theorem.

Answer the following with complete definitions or statements or short proofs. Each is worth 5 points.

6. Prove that $\{[a, b) \mid a < b\}$ is a basis for a topology on $\mathbb{R}$.
7. Give the definition of a quotient map.
8. Prove that there is no continuous map $f : \mathbb{R} \to \{0, 1\}$ that is surjective.
9. Does a connected space need to be path connected?
10. State the Intermediate Value Theorem.
11. Describe the one-point compactification of $(0, 1) \cup (2, 3)$.
12. State the Urysohn Lemma.
13. State the Tietze Extension Theorem.
14. Give an example of a space that is Hausdorff but not regular.
15. Is $\mathbb{R}^\omega$ connected in the uniform topology?