Second Semester Algebra Exam **January 2018**

Answer four problems, and on the list below circle the problems you wish to have graded. Do not circle more than four problems. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

Grade: 1 2 3 4 5 6

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1. Show that the ring

$$\mathbb{Z}[\omega] = \{a + b\omega \in \mathbb{C} \mid a,\ b \in \mathbb{Z}\} \qquad \omega = \frac{1 + i\sqrt{3}}{2}$$

is a Euclidean domain.

2. Let R be a commutative ring with identity. A subset $S \subseteq R$ is *multiplicative* if it is nonempty and is closed under multiplication. Show that if S is a multiplicative subset of R not containing 0, there is a prime ideal $\mathfrak{p} \subset R$ such that $\mathfrak{p} \cap S = \emptyset$ (First show using Zorn's lemma that the set of ideals $I \subset R$ such that $I \cap S = \emptyset$ has a maximal element. Then show, using an argument by contradiction that any such maximal element is prime).

3. Prove the following form of Eisenstein's criterion: Let R be an integral domain and $P \subset R$ a prime ideal. Suppose

$$f(X) = X^d + a_{d-1}X^{d-1} + \cdots + a_1 X + a_0 \in R[X]$$

is a monic polynomial such that $a_i \in P$ for $i < d$ and $a_0 \notin P^2$. Then $f(X)$ is irreducible in $R[X]$.

4. Let R be an integral domain. (i) Define what it means for an element of R to be *irreducible*. (ii) Define what it means for an element of R to be *prime*. (iii) Show that a prime element is irreducible.

5. Suppose R is a commutative ring with identity, M and N are R-modules and $f : M \to N$ is an R-module homomorphism. Let $I \subset R$ be an ideal. (i) (5 points) Show that there is a homomorphism $\bar{f} : M/IM \to N/IN$ such that

$$\bar{f}(m + IM) = f(m) + IN$$

for all $m \in M$. (ii) (5 points) Show that if I is nilpotent (i.e. $I^k = 0$ for some $k > 0$) and $\bar{f}$ is surjective then f is surjective.

6. Find representatives of every similarity class of 6×6 matrices with rational coefficients whose minimal polynomial is $(x + 1)^2(x - 1)$.