Second Semester Algebra Exam   **August 16, 2018**

Answer four problems. You should indicate which problems you wish to have graded. Write your solutions clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Let R be a commutative ring with 1 and let I, J, P be ideals in R such that $I \cap J \subset P$.

   (a) Prove that if P is a prime ideal then either $I \subset P$ or $J \subset P$.

   (b) Give an example which shows that if P is not prime then the conclusion above may not hold.

2. Prove that the ring $\mathbb{Z}[\sqrt{-2}] = \{a + b\sqrt{-2} : a, b \in \mathbb{Z}\}$ is a Euclidean domain with respect to the norm defined by $\mathrm{N}(a + b\sqrt{-2}) = a^2 + 2b^2$ for $a, b \in \mathbb{Z}$.

3. Prove that the following polynomials are irreducible over $\mathbb{Q}$. Cite any theorems that you use.

   (a) $X^3 - 5X^2 + X + 2$

   (b) $X^4 + 33X^2 - 6$

   (c) $X^4 + 2X^3 - 14X^2 + 31X - 7$

4. Let R be a ring with 1.

   (a) State the First Isomorphism Theorem for R-modules.

   (b) Let $A_1, A_2, \ldots, A_n$ be R-modules and for $1 \leq i \leq n$ let B_i be a submodule of A_i. Prove that

   $$(A_1 \times \cdots \times A_n)/(B_1 \times \cdots \times B_n) \cong (A_1/B_1) \times \cdots \times (A_n/B_n).$$

5. State the classification theorems for finitely generated modules over a PID. You should give both the invariant factor and the elementary divisor decompositions. Be sure to include the uniqueness statements.

6. Let L/K be an algebraic field extension and let R be a subring of L such that $K \subset R$. Prove that R is a field.