First-Semester Algebra Exam **January 4, 2018**

Answer four problems. You should indicate which problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

1. Give an example of each, with a few words of explanation.
   (a) An abelian simple group.
   (b) A nonabelian simple group.
   (c) A nilpotent group which isn't a p-group.
   (d) A finite group which is neither solvable nor simple.

2. Let G act on the set A and let $a \in A$. Prove that $|G \cdot a| = |G : G_a|$. (Note that $G \cdot a$ is the orbit containing a, while G_a is the stabilizer of a in G.)

3. Let G be a finite abelian group of order n and let k be an integer which is relatively prime to n. Define $\tau : G \to G$ by $\tau(g) = g^k$. Prove that $\tau \in \mathrm{Aut}(G)$.

4. Let G be a finite group and let H and K be subgroups of G. Recall that $HK = \{xy : x \in H,\, y \in K\}$.
   (a) Prove the following formula for the cardinality of the set HK:
   $$|HK| = \frac{|H| \cdot |K|}{|H \cap K|}.$$
   (b) Give an example in which $|HK|$ does not divide $|G|$.

5. Let G be a group and let A be an abelian group.
   (a) Define the commutator subgroup $G' = [G, G]$ of G.
   (b) Prove that if $\phi : G \to A$ is a homomorphism then there is a unique homomorphism $\overline{\phi} : G/G' \to A$ such that $\phi = \overline{\phi} \circ \pi$, where $\pi : G \to G/G'$ is the projection map.

6. Determine all groups of order 99 up to isomorphism.