Numerical Linear Algebra Exam — January, 2018
Do **4** (four) problems

1. Assume $A \in \mathbb{R}^{m,n}$ with $m \geq n$, $\text{rank}(A) = n$ and $b \in \mathbb{R}^n$.
   (a) Define the least squares solution to $Ax = b$.
   (b) Derive the normal equations for the least squares problem.
   (c) Prove that $A^T A$ is invertible.
   (d) Prove that the unique solution to the least squares problem is $(A^T A)^{-1} A^T b$.
   (e) Describe how to solve the least squares problem using the QR decomposition of A.

2. Define a normal matrix and prove that the following are equivalent.
   (a) A is normal.
   (b) $\|Ax\|_2 = \|A^* x\|_2$ for every x.
   (c) A is unitarily diagonalizable.

3. Assume $A \in \mathbb{R}^{m,m}$
   (a) Prove that $\langle x, y \rangle_A = x^* A y$ is an inner product on $\mathbb{R}^m$ if and only if A is symmetric and positive definite
   (b) Assume now that A is symmetric and positive definite. If x_* is the solution to $Ax = b$ and $\{p_1, \ldots, p_m\}$ is an orthonormal basis for $\mathbb{R}^m$ with respect to $\langle \ , \ \rangle_A$ and $x_* = \sum c_i p_i$, give a formula for the c_i.

4. If $q_1, \ldots q_n$ is an orthonormal basis for the subspace $V \subset \mathbb{C}^m$ with $m > n$, prove that the orthogonal projector onto V is QQ^*, where Q is the matrix whose columns are the q_j.

5. Assume $A \in \mathbb{C}^{m,m}$
   (a) Show that A has a Schur decomposition.
   (b) If A has a collection of m linearly dependent eigenvectors, show that A is diagonalizable.