Numerical Analysis Qualifying Exam – May, 2018

Do all five (5) problems

1. Consider

$$y' = 4ty; \quad t \in [0,1]; \quad y(0) = 2, \tag{1}$$

which has solution $Y(t) = 2e^{2t^2}$.

   (a) Using the Euler method error estimate find the value of h required to ensure that the Euler method solution w_i of (1) satisfies $|Y(t_i) - w_i| \leq 0.1$ for all i.

   (b) Derive the Taylor method of order 2 for (1).

2. Let $g \in C^2([a,b])$ and $p \in (a,b)$ with $g(p) = p, g'(p) = 0, g''(p) \neq 0$.

   (a) Show there is an $\epsilon > 0$ so that for all $x \in [p-\epsilon, p+\epsilon]$, we have $g^n(x) \to p$ as $n \to \infty$.

   (b) With ϵ as in part(a), show that for all $x \in [p-\epsilon, p+\epsilon]$, we have $|g(x) - p| \leq M|x-p|^2$ where $M = \max\{|g''(x)| \colon |x-p| \leq \epsilon\}/2$.

3. Find the second order (i.e. quadratic) least squares approximation to the function $f(x) = x^4$ on the interval $[-1,1]$ with respect to the weight $w(x) \equiv 1$ using the fact that the first three orthonormal polynomials on $[-1,1]$ with respect to this weight are

$$\varphi_0(x) = \sqrt{1/2}, \quad \varphi_1(x) = \sqrt{3/2}\, x, \text{ and } \varphi_2(x) = \sqrt{5/8}(3x^2 - 1).$$

4. (a) Assume $N(h)$ is the computed approximation for M for each $h > 0$ and

$$M = N(h) + C_1 h + C_2 h^2 + C_3 h^3 + \dots.$$

   Use the values $N(h), N(h/3)$, and $N(h/9)$ to produce a $O(h^3)$ approximation to M.

   (b) Taylor's formula yields the following.

$$f'(x_0) = \frac{1}{h}(f(x_0+h) - f(x_0)) - \frac{h}{2}f''(x_0) - \frac{h^2}{6}f'''(x_0) + O(h^3).$$

   Use this with the extrapolation of part (a) to derive an $O(h^3)$ formula for $f'(x_0)$.

5. Find a, b, c so that the quadrature formula

$$\int_0^2 f(x)\, dx = af(0) + bf(1) + cf(2)$$

has degree of precision as large as possible, and show it is no larger than your answer.