First-year Analysis Examination
Part One
January 2018

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $(a_n : n \geqslant 0)$ be a bounded real sequence. For each $N \geqslant 0$ define $\bar{a}_N = \sup\{a_n : n \geqslant N\}$ and let $s = \inf\{\bar{a}_N : N \geqslant 0\}$. Prove:
(i) if $s < t$ then $a_n < t$ eventually, in the sense $(\exists N)(\forall n \geqslant N)(a_n < t)$;
(ii) if $s > r$ then $a_n > r$ frequently, in the sense $(\forall N)(\exists n \geqslant N)(a_n > r)$.

2. Let X be a metric space. Assume that there exists $\delta > 0$ such that the distance between distinct points of X is always δ or greater. Determine precisely which subsets of X are open, which closed, which compact and which connected.

3. Let $f : X \to \mathbb{R}$ have the property that if a is any real number then $U(a) = \{x : f(x) < a\}$ is an open subset of the *compact* metric space X.
(i) Prove that f is bounded above on X.
(ii) Prove that $s = \sup\{f(x) : x \in X\}$ is a value of f.
For (ii): suppose s not a value and consider the function $F = 1/(s - f)$.

4. Let $f : X \to Y$ be a bijection between metric spaces; assume that Y is complete and that f^{-1} is continuous.
(i) Show that if f is uniformly continuous then X is complete.
(ii) Show that if f is only continuous then X can fail to be complete.

5. Let $f : (a, b) \to \mathbb{R}$ be differentiable and assume that its derivative f' is bounded. Prove that the right-hand limit $\lim_{t \to a+} f(t)$ exists.
