First-year Analysis Examination
Part Two
May 2018

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. (i) Let $g_n \to g$ and $h_n \to h$ uniformly on X. Prove that if g and h are bounded, then $g_n h_n \to gh$ uniformly on X.
(ii) Let $f : [a, b] \to \mathbb{R}$ be continuous. By considering the square-root or otherwise, prove that if f is non-negative then there exists a sequence of *non-negative* polynomials converging uniformly to f on $[a, b]$.

2. Let $\mathcal{F}$ be an equicontinuous family of real-valued functions on the connected space X. By considering the set of points at which the family $\mathcal{F}$ is bounded, prove that if $\mathcal{F}$ is bounded at one point of X then it is bounded at all points of X.

3. $(f_n : n \geqslant 0)$ is a sequence of measurable real-valued functions. Prove that each of the following sets is measurable:
(a) the set A comprising all points ω for which the sequence of values $f_n(\omega)$ is eventually positive;
(b) the set B comprising all points ω for which the sequence of values $f_n(\omega)$ changes sign infinitely often.

4. $(\Omega, \mathcal{F}, \mu)$ is a measure space on which $(f_n : n \geqslant 0)$ is a sequence of Lebesgue-measurable functions converging uniformly to f. Prove that if $\mu(\Omega) < \infty$ then it follows that f is integrable and $\int_\Omega f_n \mathrm{d}\mu \to \int_\Omega f \mathrm{d}\mu$. Show by example that if $\mu(\Omega) = \infty$ then the same conclusion may not follow.

5. Let $(f_n : n \geqslant 0)$ be a uniformly-bounded sequence of Riemann-integrable functions on $[a, b]$. If $f_n \to 0$ pointwise, does it follow that $\int_a^b f_n \to 0$? Proof or counterexample required.
