First-year Analysis Examination
Part Two
January 2018

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $f : [0,1] \to \mathbb{R}$ be Riemann-integrable. Prove that if the Riemann integral $\int_0^1 f$ is nonzero then there exist $\delta > 0$ and a nonempty open interval $I \subseteq [0,1]$ such that $|f| \geqslant \delta$ throughout I.

2. Let $f_n(t) = t^n(1-t)^n$ whenever $0 \leqslant t \leqslant 1$ and n is a positive integer. Does the sequence $(f_n : n > 0)$ converge on $[0,1]$ pointwise? Uniformly? Justify.

3. Let $f : [0,1] \to \mathbb{R}$ be continuous. Assume that $\int_0^1 f(t)e^{-nt}\mathrm{d}t = 0$ whenever n is a non-negative integer. Does it follow that f is identically zero? Does the answer change if the exponent $-nt$ is replaced by $+nt$? By $2nt$? Justify.

4. Given that $(f_n : n \geqslant 0)$ is a sequence of measurable real-valued functions, explain why each of the following sets is (or is not) measurable:
   (i) $X = \{\omega : f_n(\omega) \to \infty \text{ as } n \to \infty\}$;
   (ii) $Y = \{\omega : f_n(\omega)^2 < f_n(\omega) \text{ for each } n \geqslant 0\}$;
   (iii) $Z = \{\omega : \sum_{n=0}^{\infty} f_n(\omega) \text{ converges absolutely}\}$.

5. Let $(f_n : n \geqslant 1)$ be a sequence of continuous functions from $[0,1]$ to $[0,1]$ and assume that this sequence converges to f pointwise. True or false (proof or counterexample)?
   (i) f is Riemann-integrable and (Riemann integrals) $\int_0^1 f_n \to \int_0^1 f$;
   (ii) f is Lebesgue-integrable and (Lebesgue integrals) $\int_0^1 f_n \to \int_0^1 f$.
