First-year Analysis Examination
Part Two
August 2018

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let the sequence $(s_n)_{n=1}^{\infty}$ in $[0,1]$ be *uniformly distributed* in the sense that if $0 \leqslant a \leqslant b \leqslant 1$ then

$$\lim_{n\to\infty} \frac{\#\{k \leqslant n : s_k \in [a,b]\}}{n} = b - a.$$

Let $f : [0,1] \to \mathbb{R}$ and prove that

$$\lim_{n\to\infty} \frac{f(s_1) + \cdots + f(s_n)}{n} = \int_0^1 f(t)\mathrm{d}t$$

in each of the following cases:
(i) f is a step function (a finite linear combination of indicators of intervals);
(ii) f is Riemann-integrable.

2. Fix $a \in \mathbb{R}$ and for each integer $n > 0$ write $f_n(t) = n^a t(1-t^2)^n$ whenever $0 \leqslant t \leqslant 1$.
(i) Show that $(f_n)_{n=1}^{\infty}$ converges pointwise on $[0,1]$; say to f.
(ii) For which values of a does $(f_n)_{n=1}^{\infty}$ converge uniformly on $[0,1]$? Justify.
(iii) For which values of a is it true that $\int_0^1 f_n \to \int_0^1 f$? Justify.

3. Let $f : [1,\infty) \to \mathbb{R}$ be continuous and satisfy $\lim_{t\to\infty} f(t) = A \in \mathbb{R}$. Prove that there is a sequence $(p_n)_{n=0}^{\infty}$ of polynomials such that $p_n(1/t)$ converges to $f(t)$ uniformly for $t \geqslant 1$.

4. Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued measurable functions. Prove that each of the following sets is measurable:
(i) $B = \{\omega : (f_n(\omega))_{n=1}^{\infty} \text{ has no biggest term}\}$;
(ii) $C = \{\omega : \cos(f_n(\omega)) > 0 \text{ for each } n > 0\}$;
(iii) $D = \{\omega : (f_n(\omega))_{n=1}^{\infty} \text{ does not converge to a rational number}\}$.

5. State the *Monotone Convergence Theorem.* Use it to prove the following: let $(f_n)_{n=1}^{\infty}$ be a sequence of non-negative integrable functions with pointwise limit f; if $\int f_n \mathrm{d}\mu \leqslant M < \infty$ for each n then f is integrable and $\int f \mathrm{d}\mu \leqslant M$.
