First-year Analysis Examination
Part One
May 2018

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $f : [0,1] \to [0,1]$ be a (weakly) increasing function. Prove that

$$\alpha := \inf\{t \in [0,1] : f(t) \leqslant t\}$$

exists and is fixed by f. Show further that α is the *smallest* fixed point of f.

2. Let M be a metric space, let $U \subseteq M$ be open and let $A \subseteq M$ be arbitrary. Prove that $\overline{A} \cap U \subseteq \overline{A \cap U}$. Can this inclusion be strict? Explain.

3. Let $(a_n : n \geqslant 0)$ be a sequence of real numbers. Assume that there exists $\lambda \in (0,1)$ such that $|a_{n+1} - a_n| \leqslant \lambda |a_n - a_{n-1}|$ for each positive integer n. Prove that the sequence $(a_n : n \geqslant 0)$ converges.

4. Let $f : X \to Y$ be a continuous bijection between metric spaces. For each of the following statements, decide whether true or false, giving proof or counterexample as appropriate:
(i) if X is compact, then $f^{-1} : Y \to X$ is continuous;
(ii) if Y is compact, then $f^{-1} : Y \to X$ is continuous.

5. Let the real-valued function f be differentiable on the open interval $(0,1)$. Show that if its derivative f' is bounded, then f extends to a continuous function on the closed interval [0, 1].
