First-year Analysis Examination
Part One
August 2018

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $(a_n)_{n=0}^{\infty}$ be a sequence of positive real numbers. Prove that

$$1/\limsup a_n = \liminf(1/a_n)$$

in the sense that if one side lies in $(0, \infty)$ then so does the other and the two sides are equal.

2. Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers and $A = \{a_n : n \geqslant 0\}$ its set of values. For p a real number, consider the two implications:
(i) if p is a limit point of A then p is the limit of a subsequence of $(a_n)_{n=0}^{\infty}$;
(ii) if p is the limit of a subsequence of $(a_n)_{n=0}^{\infty}$ then p is a limit point of A.
One of these is true and the other false; give a proof for the true and a counterexample for the false.

3. Let X be a *compact* metric space and let $f : X \to X$ be a continuous function. Prove that if f has no fixed points, then there exists $a > 0$ such that $d(x, f(x)) \geqslant a$ whenever $x \in X$. Show by example that the *compact* hypothesis cannot be dropped.

4. (i) Prove that if c is any real number then the following subset of the Euclidean plane is connected

$$\{(x, y) : (x = \pm 1) \ \text{ or } \ (|x| < 1 \ \text{ and } \ y \geqslant c)\}.$$

(ii) Let $C_0 \supseteq C_1 \supseteq \dots$ be a decreasing sequence of closed, connected subsets of a metric space. If the intersection $\cap_{n \geqslant 0} C_n$ is nonempty, must it be connected? Proof or counterexample.

5. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and assume that $f'(t) \to 0$ as the real number t tends to infinity. Show that if $f(n) \to \ell$ as the integer n tends to infinity, then $f(t) \to \ell$ as the real number t tends to infinity.
