1st Year Topology Exam, May 10th 2013

Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.

1. Prove that the spaces $[0, 1]$ and $[0, 1)$ are not homeomorphic.

2. Let X be a connected normal topological space having more than one point. Prove that X is uncountable

3. Let $f : X \to Y$ be a continuous map to a Hausdorff space. Prove that the graph of f, $G = \{(x, f(x)) \mid x \in X\}$, is a closed subset of $X \times Y$.

4. (a) Let X be a compact space. Show that for every topological imbedding $f : X \to Y$ into a Hausdorff space the image $f(X)$ is closed in Y.

(b) Suppose that a normal topological space X has the property that for every topological imbedding $f : X \to Y$, the image $f(X)$ is closed in Y. Does it follow that X is compact?

5. Is every closed subset A of a separable space X separable itself if (a) $X = \mathbb{R}$? (b) $X = \mathbb{R} \times \mathbb{R}$? (c) $X = \mathbb{R}_\ell$? (d) $X = \mathbb{R}_\ell \times \mathbb{R}_\ell$?

6. Does there exist a covering map $p : \mathbb{R}^2 \to \mathbb{RP}^2$ from the Euclidean plane to the projective plane?

Answer the following with complete definitions or statements or short proofs.

7. State the Tietze Extension Theorem.

8. Is the space $\mathbb{R}^\omega$ connected in the uniform topology?

9. Does there exist a continuous surjective map from the 2-sphere S^2 to the punctured square $([-1, 1] \times [-1, 1]) \setminus \{(0, 0)\}$?

10. Is every connected space path connected?

11. What is a basis of a topology? What is a subbasis?

12. State the Baire Category Theorem.

13. Is the unit circle $S^1 = \{x \in \mathbb{R}^2 \mid \|x\| = 1\}$ a
(a) retract of $\mathbb{R}^2$? (b) retract of $\mathbb{R}^2 \setminus \{(2, 0)\}$? (c) deformation retract of $\mathbb{R}^2 \setminus \{(0, 0)\}$? (d) deformation retract of $\mathbb{R}^2 \setminus \{(0, 0), (2, 0)\}$? (e) retract of $\mathbb{R}^2 \setminus \{(0, 0), (2, 0)\}$?

14. State the Brouwer fixed point theorem. Does every continuous map $f : [0, 1] \times [0, 1] \to [0, 1) \times [0, 1)$ have a fixed point?

15. Can the space of irrationals in the subspace topology be presented as a countable union of nowhere dense subsets?