# 1st Year Topology Exam

January 4th, 2013

**Work the following problems and show all work. Support all statements to the best of your ability. Work each problem on a separate sheet of paper.**

1. Show that the set of functions $f : \mathbb{Z} \to \mathbb{Z}$ that are eventually constant is countable.

2. Let $X$ be a connected metric space having more than one point. Can $X$ be countable?

3. Show that every continuous map $f : [0,1] \to [0,1]$ has a fixed point. Is this true for for continuous maps
(a) $f : (0,1) \to (0,1)$?
(b) $f : [0,1] \to [0,1)$?

4. Let $A$ be a proper subset of $X$ and let $B$ be a proper subset of $Y$. If $X$ and $Y$ are connected, show that $(X \times Y) - (A \times B)$ is connected.

5. Let $f : X \to X$ be a map of a complete metric space to itself that satisfies the following condition: There exists $\lambda < 1$ such that $d(f(x), f(y)) \leq \lambda d(x,y)$. Prove that $f$ is continuous. Show that $f$ has a fixed point and the fixed point is unique.

**Answer the following with complete definitions or statements or short proofs.**

6. State the Intermediate Value Theorem.

7. Does there exist a continuous surjective map from the 2-sphere $S^2$ to the interval $(0,1)$?

8. Does there exist a continuous injective map from the 2-sphere $S^2$ to the interval $(0,1)$?

9. State the Cantor-Schroeder-Bernstein Theorem.

10. Is every connected space path connected?

11. What is a basis of a topology? Does the set of all half-open intervals $\{(a,b], [b,c) \mid a < b,\ c < d\}$ form a basis of a topology on $\mathbb{R}$?

12. State the Extreme Value Theorem.

13. Is $\mathbb{R}^\omega$ connected in the uniform topology?

14. Is the subspace $X = \{(x,y) \in \mathbb{R}^2 \mid y = \frac{1}{x}\}$ compact? locally compact?

15. Describe all connected subsets of $\mathbb{R}$.