FIRST-YEAR EXAM - SECOND-SEMESTER TOPOLOGY

Answer all questions and work all problems. Each problem is worth the points allotted.

Problem 1. (5 points) Let $n > 1$. Suppose that $f, g : X \to S^n$ are continuous such that $f(x)$ and $g(x)$ are not antipodal for all $x \in X$. Show that f and g are homotopic.

Problem 2. (5 points) Show that $\pi_1(S^n, 1) = 0$ for all $n > 1$.

Problem 3. (5 points) Show that $\pi_1(S^1, 1) = \mathbb{Z}$.

Problem 4. (5 points) Show that the disk, $D^2 = \{z \in \mathbb{C} \bigm| ||z|| \leq 1\}$ has the fixed point property.

Problem 5. (5 points) Consider the matrix $M = \begin{bmatrix} 2 & 3 \\ 1 & 1 \end{bmatrix}$. This represents a group homomorphism $M : Z^2 \to Z^2$. Show that there is a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ such that $f_* = M$ where $f_* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$ is the homomorphism induced by f on the fundamental group.

Problem 6. (5 points) Consider three loops $f, g, h : [0, 1] \to (X, x_0)$. Show that $(f \circ g) \circ h$ is homotopic to $f \circ (g \circ h)$ where $\circ$ denotes concatenation of loops.

Problem 7. (5 points) Let $n > 1$. Let $P^n = S^n/D$ where D identifies x with its antipode for all $x \in S^n$. Show that $\pi_1(P^n, x_0) \cong \mathbb{Z}_2$.

Problem 8. (5 points) Describe a CW-complex X such that $\pi_1(X, x_0) \cong \mathbb{Z}_3$.

Problem 9. (5 points) Suppose that $\pi_1(X, x_0) \cong \mathbb{Z}_2$ and that $\pi_1(Y, y_0) \cong \mathbb{Z}_5$. Show that $\pi_1\left(X \times Y, (x_0, y_0)\right) \cong \mathbb{Z}_{10}$.

Problem 10. (5 points) Suppose that X and Y are connected. Show that $X \times Y$ is connected.

Problem 11. (5 points) Let $n > 1$. Let $A \subset \mathbb{R}^n$ be a countable set. Show that $\mathbb{R}^n \setminus A$ is arcwise connected.

Problem 12. (5 points) Let $X = S^1 \vee S^1$. Show that $\pi_1(X) \cong \mathbb{Z} * \mathbb{Z}$.

Problem 13. (40 points) State the following theorems.

Seifert-van Kampen Theorem.

The Urysohn Metrization Theorem

The Urysohn Lemma

The Tietze Extension Theorem

The Brouwer Fixed Point Theorem

The Fundamental Theorem of Algebra

The Jordan Curve Theorem

The Arcwise Connectedness Theorem

The Hahn-Mazurkiewicz Theorem