FIRST-YEAR PH.D. EXAM - SECOND-SEMESTER TOPOLOGY

Answer all questions and work all problems. Each problem is worth the points allotted.

Problem 1. (5 points) Suppose that X is a regular Lindelöf space. Show that X is normal.

Problem 2. (5 points) Show that there is a continuous $f : [0,1] \to [0,1]^2$ such that f is onto.

Problem 3. (5 points) Suppose that $f, g : X \to S^n$ are continuous maps such that for all $x \in X$, $f(x)$ and $g(x)$ are not antipodal. Show that f and g are homotopic.

Problem 4. (5 points) Show that the Sorgenfrey line R is regular and Lindelöf.

Problem 5. (5 points) Let (X, x_0) be a pointed space. The **trivial loop** is the function $f : [0,1] \to (X, x_0)$ such that $f(t) \equiv x_0$ for all $t \in [0,1]$. Let $f : [0,1] \to (X, x_0)$ be a loop. Define f^{-1} by $f^{-1}(t) = f(1-t)$. Show that f^{-1} is a loop. Show that $f \circ f^{-1}$ is homotopic to the trivial loop.

Problem 6. (5 points) Consider three loops $f, g, h : [0,1] \to (X, x_0)$. Show that $(f \circ g) \circ h$ is homotopic to $f \circ (g \circ h)$.

Problem 7. (5 points) Let (X, x_0) be a pointed space and let $\pi_1(X, x_0)$ be the collection of homotopy classes of loops. Show that $\pi_1(X, x_0)$ is a group.

(1) Let $\alpha = [a]$ and $\beta = [b]$ be elements of $\pi_1(X, x_0)$. Define $\alpha \cdot \beta = [a \circ b]$. Show that $\alpha \cdot \beta$ is well-defined.
(2) Show that the operation $\alpha \cdot \beta$ is associative.
(3) Show that under the operation $\alpha \cdot \beta$, the homotopy class of the trivial loop is the identity.
(4) Show that under the operation $\alpha \cdot \beta$, the homotopy class of a^{-1} is the inverse of α.

Problem 8. (5 points) Show that $\pi_1(S^1, 1) = \mathbb{Z}$.

Problem 9. (5 points) Show that $\pi_1(S^n, 1) = 1$ for all $n > 1$.

Problem 10. (5 points) Let $n > 1$. Let $P^n = S^n/D$ for where D identifies x with $-x$ for all $x \in S^n$. Show that $\pi_1(P^n, x_0) = \mathbb{Z}_2$.

Problem 11. (5 points) Consider the matrix $M = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. This represents a group homomorphism $M : Z^2 \to Z^2$. Show that there is a map $f : \mathbb{T}^2 \to \mathbb{T}^2$ such that $f_* = M$ where $f_* : \pi_1(\mathbb{T}^2) \to \pi_1(\mathbb{T}^2)$ is the homomorphism induced by f on the fundamental group.

Problem 12. (5 points) Let $G =< a, b, c \mid a \cdot b^{-1} \cdot c >$ be a finitely presented group with 3 generators and 1 relation.

(a) Give a space X whose fundamental group is the free group on three generators.
(a) Describe a space Y whose fundamental group is the group G.

Problem 13. (40 points) State the following theorems.

Seifert-van Kampen Theorem.

The Urysohn Metrization Theorem

The Urysohn Lemma

The Tietze Extension Theorem

The Brouwer Fixed Point Theorem

The Fundamental Theorem of Algebra

The Jordan Curve Theorem

The Arcwise Connectedness Theorem

The Hahn-Mazurkiewicz Theorem