MTG5316, FIRST YEAR EXAM, JANUARY, 2016

Part 1. Complete Proofs (7 items, 10 points each). Work each problem on a sepaprate sheet of paper with your name on the sheet. Support all statements to the best of your ability.

1. Let X be a topological space. Prove that if X is path connected, then X is connected.

2. Let X be a topological space, let $A \subset X$, and let $B \subset X$. Let $\overline{A}$ denote the closure of A. Prove that

$$\overline{A} \cup \overline{B} = \overline{A \cup B}.$$

3. Let I be an abitrary index set, and let

$$X = \prod_{i \in I} X_i.$$

Prove that if each X_i is connected, then X is connected. You may use the fact that a finite product of connected spaces is connected, without proving this fact.

4. Let (X, d) be a metric space. Prove that if X is sequentially compact, then X is compact.

Recall the definition: X is sequentially compact if and only if every sequence of points in X has a subsequence which converges to a point of X.

5. Let (X, d) be a nonempty complete metric space with no isolated points. Prove that X is uncountable.

Recall the definition: A point x is an isolated point of X if and only if the set $\{x\}$ is open.

6. Prove that a compact Hausdorff (T_2) space is normal.

7. Define an equivalence relation $\sim$ on $X = [0, 1]$ by $x \sim x$ for all x, $0 \sim 1$, and $1 \sim 0$. Prove that the quotient space $X/\sim$ is homeomorphic to the circle S^1.

Part 2. Answer the following with complete definitions or statements or short proofs (6 items, 5 points each).

8. Complete the following definition: Let $(X, \mathcal{T})$ be topological space, and let A be a subset of X. The relative topology or subspace topology on A is defined as follows:

9. Complete the following definition: Let E be a set, and let $\mathcal{B}$ be a collection of subsets of E. We say that $\mathcal{B}$ is a basis for a topology on E if and only if

10. Complete the following definition: Let I be any set, and let $\{(X_i, \mathcal{T}_i) : i \in I\}$ be an indexed family of topological spaces. Let

$$X = \prod_{i \in I} X_i.$$

The product topology on X is defined as follows.

11. State the Extreme Value Theorem.

12. State the Contraction Mapping Theorem. (This theorem is also known as the Banach Fixed Point Theorem.)

13. Let X denote the set of real numbers with the finite complement topology. Let Y denote the set of real numbers with the usual topology. Does there exist a continuous, surjective (onto) function $f : X \to Y$?