Numerical Linear Algebra Exam January, 2016

Do **4** (four) problems

1. Prove that every square matrix A has a Schur factorization.

2. Given a matrix $A \in \mathbb{C}^{m\times n}$, let

$$B = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad \text{and} \quad C = A^*A$$

   (a) Show that the singular values of A are the absolute values of eigenvalues of B.

   (b) Show that the singular values of A are the square roots of eigenvalues of C.

   (c) Assume now that A is square and invertible. Compute the two-norm condition numbers of B and C in terms of the two-norm condition number of A.

   (d) If A has condition number bigger than one, which has the larger condition number, B or C?

3. (a) Compute $\det(\lambda I + uv^*)$ when $\lambda \in \mathbb{C}$, I is the $m\times m$ identity matrix, and $u, v \in \mathbb{C}^m$.

   (b) Prove necessary and sufficient conditions for $I + uv^*$ to be nonsingular and when it is, give a formula for its inverse.

4. (a) Given Cholesky decomposition of the Hermitian positive definite matrix $A = R^*R$, prove that $\|R\|_2 = \|R^*\|_2 = \|A\|_2^{1/2}$.

   (b) Now assume that B is a matrix that can be expressed as $B = T^*T$ for some upper triangular matrix T. Show that B is Hermitian and positive semi-definite, i.e. $x^*Bx \geq 0$ for all x.

5. Let $\{q_1, q_2, \ldots, q_n\}$ be an orthonormal subset of $\mathbb{C}^m$. Show that

$$P = \sum_{i=1}^{n} q_i q_i^*$$

   is an orthogonal projector with range equal to the span of $\{q_1, q_2, \ldots, q_n\}$.