Numerical Linear Algebra Exam – Part I – January, 2015
Do **4** (four) problems

1. Assume $A \in \mathbb{R}^{m,n}$ with $m \geq n$, $\mathrm{rank}(A) = n$ and $b \in \mathbb{R}^n$.
   (a) Define the least squares solution to $Ax = b$.
   (b) Derive the normal equations for the least squares problem.
   (c) Prove that $A^T A$ is invertible.
   (d) Prove that the unique solution to the least squares problem is $(A^T A)^{-1} A^T b$.
   (e) Describe how to solve the least squares problem using the QR decomposition of A.

2. (a) Compute $\det(\lambda I + uv^*)$ when $\lambda \in \mathbb{C}$, I is the $m \times m$ identity matrix, and $u, v \in \mathbb{C}^m$.
   (b) Prove necessary and sufficient conditions for $I + uv^*$ to be nonsingular and when it is, give a formula for its inverse.

3. (a) If P is a projector, prove that $\mathrm{null}(P) \cap \mathrm{range}(P) = \emptyset$ and $\mathrm{null}(P) = \mathrm{range}(I - P)$.
   (b) Prove that P is an orthogonal projector if and only if it is Hermitian.
   (c) If $q_1, \ldots q_n$ is an orthonornal basis for the subspace $V \subset \mathbb{C}^m$ with $m > n$, prove that the orthognal projector onto V is QQ^*, where Q is the matrix whose columns are the q_j.

4. (a) Prove that $\|A\|_F = \mathrm{trace}(A^* A)^{1/2}$.
   (b) Prove that $\|A\|_F = (\sum \sigma_i^2)^{1/2}$, where $\{\sigma_i\}$ are the singular values of A counted with multiplicity.
   (c) If both A and U are in $\mathbb{C}^{m,m}$ and U is unitary, prove that $\|UA\|_F = \|AU\|_F = \|A\|_F$.

5. Define a normal matrix and prove that the following are equivalent.
   (a) A is normal.
   (b) A is unitarily diagonalizable.
   (c) $\|A\|_F = (\sum |\lambda_i|^2)^{1/2}$, where $\{\lambda_i\}$ are the eigenvalues of A counted with multiplicity.