Numerical Linear Algebra Exam August, 2016

Do **4** (four) problems

1. Define a normal matrix and prove that the following are equivalent.
   (a) A is normal.
   (b) A is unitarily diagonalizable.
   (c) $\|A\|_F = (\sum |\lambda_i|^2)^{1/2}$, where $\{\lambda_i\}$ are the eigenvalues of A counted with multiplicity.

2. Let $\kappa_2(A)$ be the two-norm condition number of the square, non-singular A.
   (a) Prove that
   $$\kappa_2(A) = \frac{\sigma_1}{\sigma_m}.$$
   where σ_1 and σ_m are the largest and smallest singular values of A, respectively.
   (b) Prove or disprove: If $A = QBQ^*$ with Q unitary, then $\kappa_2(A) = \kappa_2(B)$.
   (c) Prove or disprove: If $A = CBC^{-1}$, then $\kappa_2(A) = \kappa_2(B)$.

3. Assume $A \in \mathbb{R}^{m,n}$ with $m \geq n$, $\text{rank}(A) = n$ and $b \in \mathbb{R}^n$.
   (a) Define the least squares solution to $Ax = b$.
   (b) Derive the normal equations for the least squares problem.
   (c) Prove that A^TA is invertible.
   (d) Prove that the unique solution to the least squares problem is $(A^TA)^{-1}A^Tb$.
   (e) Describe how to solve the least squares problem using the QR decomposition of A.

4. (a) Prove that P is an orthogonal projector if and only if it is Hermitian.
   (b) Let $\{q_1, q_2, \ldots, q_n\}$ be an orthonormal subset of $\mathbb{C}^m$. Show that
   $$P = \sum_{i=1}^{n} q_i q_i^*$$
   is an orthogonal projector with range equal to the span of $\{q_1, q_2, \ldots, q_n\}$

5. Assume $A \in \mathbb{R}^{m,m}$
   (a) Prove that $\langle x, y \rangle_A = x^TAy$ is an inner product on $\mathbb{R}^m$ if and only if A is symmetric and positive definite
   (b) Assume now that A is symmetric and positive definite. If x_* is the solution to $Ax = b$ and $\{p_1, \ldots, p_m\}$ is an orthonormal basis for $\mathbb{R}^m$ with respect to $\langle\ ,\ \rangle_A$ and $x_* = \sum c_i p_i$, give a formula for the c_i.