Numerical Analysis Qualifying Exam (January 14, 2013). *Answer any 4 questions.*

1. Suppose that $A \in \mathbb{C}^{n\times n}$ is Hermitian with λ_1 and λ_n the smallest and largest eigenvalues of A respectively. Show that
$$\lambda_1\|x\|^2 \le x^\mathsf{T} A x \le \lambda_n \|x\|^2$$
for all $x \in \mathbb{C}^n$.

2. (a) Suppose p and $q \in \mathbb{R}$ with p and q positive and $p^{-1}+q^{-1}=1$. Show that for any matrix $A \in \mathbb{C}^{m\times n}$, we have $\|A\|_p = \|A^*\|_q$, where A^* is the conjugate transpose of A.

   (b) Prove that
$$\|A\|_2^2 \le \|A\|_p\|A\|_q$$
for any $A \in \mathbb{C}^{m\times n}$ and any positive p and $q \in \mathbb{R}$ with $p^{-1}+q^{-1}=1$.

   (c) Prove that for any $p \ge 1$ and any diagonal matrix $D \in \mathbb{C}^{n\times n}$, we have
$$\|D\|_p = \max\{|d_{ii}| : 1 \le i \le n\}.$$

3. (a) Show that the eigenvalues of a projector are either 0 or 1.

   (b) Show that a projector P is orthogonal if and only if $P = P^*$.

4. Show that the element of largest magnitude in a Hermitian, positive definite matrix lies on the diagonal.

5. Suppose that $A \in \mathbb{C}^{n\times n}$ is an invertible matrix, $u \in \mathbb{C}^n$, and $v \in \mathbb{C}^n$.

   (a) Show that
$$\det(A - uv^*) = (1 - v^*A^{-1}u)\det A.$$

   (b) Suppose that A is a real diagonal matrix and $u = v$ with $u_i \ne 0$ for each i. Show that the eigenvalues of $A - uu^*$ are the roots of
$$\sum_{i=1}^{n} \frac{|u_i|^2}{a_{ii} - \lambda} = 1.$$