First-Year Analysis Examination
August 2016 Part One

Answer exactly FOUR questions. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

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1. Let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be bounded real sequences and let $(c_n)_{n=1}^{\infty}$ be the sequence

$$a_1, b_1, a_2, b_2, \ldots$$

Prove that

$$\limsup_{n\to\infty} c_n \leqslant \max(\limsup_{n\to\infty} a_n, \limsup_{n\to\infty} b_n)$$

and that if $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ converge to γ then so does $(c_n)_{n=1}^{\infty}$.

2. State the definition of a perfect set and prove that the standard Cantor set in $[0, 1]$ is a perfect set.

3. Let $f : X \to Y$ be a continuous map between metric spaces and let

$$G_f = \{(x, f(x)) : x \in X\} \subseteq X \times Y$$

be its graph. Prove that if G_f is compact then f is continuous.

4. Let f be a *uniformly* continuous real-valued function on the bounded open interval (a, b). Prove that the real limit $\lim_{t\to b-} f(t)$ exists. Show by example that we may not replace *uniformly* by *bounded* without jeopardizing this conclusion.

5. Let the real-valued function f be continuous on $[0, 2]$ and twice-differentiable on $(0, 2)$. Assume that $f(0) = 0$, that $f(1) = 1$ and that $f(2) = 2$. Prove that there exists $u \in (0, 2)$ such that $f''(u) = 0$.