First-year Analysis Examination
Part Two
August 2017

Answer FOUR questions in detail.
State carefully any results used without proof.

---

1. Let $f : [0,1] \to [0,1)$ be continuous. Decide whether it follows that

$$\int_0^1 \frac{1}{1-f(t)}\,\mathrm{d}t = \sum_{n=0}^{\infty} \int_0^1 f(t)^n \mathrm{d}t$$

giving proof or counterexample as appropriate. [Note the intervals carefully.]

2. Let $f : \mathbb{R} \to \mathbb{R}$ be nonconstant; for each positive integer n and each real t define $f_n(t) = f(nt)$. Prove that there exists no $\varepsilon > 0$ such that $(f_n)_{n=1}^{\infty}$ is equicontinuous on the interval $(-\varepsilon, \varepsilon)$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. **Either** (i) show that there exists a sequence of polynomials converging *pointwise* to f on $\mathbb{R}$ **or** (ii) show that there need not exist a sequence of polynomials converging *uniformly* to f on $\mathbb{R}$.

4. Let $(f_n)_{n=1}^{\infty}$ be a sequence of measurable real-valued functions. Prove that each of the following sets is measurable:
(i) $A = \{\omega : f_n(\omega) \to \infty \text{ as } n \to \infty\}$;
(ii) $B = \{\omega : f_n(\omega) \text{ is eventually irrational}\}$;
(iii) $C = \{\omega : f_n(\omega) > 0 \text{ for infinitely many } n\}$.

5. Let $f : [0,\infty) \to \mathbb{R}$ be Lebesgue integrable and assume that $f(t) \to 1$ as $t \to \infty$. Prove that for each positive integer n we may define

$$a_n = \frac{1}{n} \int_0^{\infty} e^{-t/n} f(t)\mathrm{d}t \in \mathbb{R}$$

and prove that $a_n \to 1$ as $n \to \infty$.
Suggestion: Say $|f(t) - 1| \leqslant \varepsilon$ whenever $t \geqslant A$ and split the integral.
