First-year Analysis Examination
Part Two
May 2017

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $(f_n)_{n=0}^{\infty}$ be a sequence of differentiable functions. Decide whether each implication is valid, giving proof or counterexample as appropriate:
(i) if $f_n \to f$ uniformly on $(-\infty, \infty)$ then $f_n^2 \to f^2$ uniformly on $(-\infty, \infty)$;
(ii) if $f_n \to f$ uniformly on $[-1, 1]$ then $\int_{-1}^{1} f_n \to \int_{-1}^{1} f$;
(iii) if $f_n \to f$ uniformly on $(-1, 1)$ then f is differentiable and $f_n' \to f'$ uniformly on $(-1, 1)$.

2. Let f be a continuous real-valued function on $[0, 1]$ and let $\alpha \in (0, \infty)$. Assume that

$$\int_0^1 t^{n\alpha} f(t) \mathrm{d}t = 0$$

for all but finitely many values of $n \in \mathbb{N}$. What conclusions can be drawn about f?

3. Let $\mathcal{F}$ be an equicontinuous family of real-valued functions on the compact metric space X. Denote by $A \subseteq X$ the set whose elements are precisely those $a \in X$ at which $\mathcal{F}$ is *bounded* in the sense that $\{f(a) : f \in \mathcal{F}\} \subseteq \mathbb{R}$ is bounded. Prove that A is both open and closed.

4. Let $(f_n)_{n=0}^{\infty}$ be a sequence of measurable real-valued functions. Decide whether each of the following sets is measurable:
(i) $\{\omega : (f_n(\omega))_{n=0}^{\infty}$ is unbounded$\}$;
(ii) $\{\omega : (f_n(\omega))_{n=0}^{\infty}$ is periodic$\}$;
(iii) $\{\omega : (f_n(\omega))_{n=0}^{\infty}$ has distinct terms$\}$.

5. Let $(\Omega, \mathcal{A}, \mu)$ be a measure space that is *finite* in the sense that $\mu(\Omega) < \infty$. Let $(f_n)_{n=0}^{\infty}$ be a sequence of non-negative measurable functions converging pointwise to f on Ω. True or false (proof or counterexample):

$$\int_{\Omega} \frac{1}{1 + f_n} \,\mathrm{d}\mu \to \int_{\Omega} \frac{1}{1 + f} \,\mathrm{d}\mu \text{ as } n \to \infty.$$
