First-year Analysis Examination
Part Two
January 2016

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $f_n : [a, b] \to \mathbb{R}$ be continuously differentiable and let f_n' converge to g uniformly on $[a, b]$. Prove that if f_n converges pointwise at a then f_n converges uniformly on $[a, b]$ to a continuously differentiable function f such that $f' = g$.

2. Let f be a real-valued function on $[a, b]$. For each of the following statements, give a proof or a counterexample, as appropriate.

(i) If f is *Riemann* integrable with strictly positive *Riemann* integral over $[a, b]$ then f is strictly positive on some nonempty open interval.

(ii) The same statement, with *Riemann* replaced by *Lebesgue* throughout.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. For $a \leqslant b$ and for N a natural number, consider the statement:

'if $\int_a^b f(t)t^{2n+1}\mathrm{d}t = 0$ for each integer $n \geqslant N$ then $f = 0$ on $[a, b]$'.

Argue the truth or falsity of this statement in the following cases:
(i) $N = 0, [a, b] = [0, 1]$; (ii) $N = 1, [a, b] = [0, 1]$; (iii) $N = 0, [a, b] = [-1, 1]$.

4. Let f_n be a sequence of measurable real-valued functions on the measurable space (Ω, Σ). Prove that each of the following sets is measurable: (i) the set P comprising all $\omega \in \Omega$ such that $f_n(\omega)$ converges to an irrational number; (ii) the set Q comprising all $\omega \in \Omega$ such that $f_n(\omega)$ converges to a rational number; (iii) the set R comprising all $\omega \in \Omega$ such that $f_n(\omega)$ converges to a real number.

5. Let f be an integrable function on the measure space (Ω, Σ, μ). Prove that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that if $A \in \Sigma$ and $\mu(A) < \delta$ then $\int_A |f| \mathrm{d}\mu < \varepsilon$.
