# First-year Analysis Examination
# Part One
# August 2017

---

Answer FOUR questions in detail.
State carefully any results used without proof.

---

1. $A$ and $B$ are nonempty sets of real numbers such that: (i) $A \cup B = \mathbb{R}$ and (ii) $a < b$ whenever $a \in A$ and $b \in B$. Prove that there exists a unique $s \in \mathbb{R}$ such that $(-\infty, s) \subseteq A$ and $(s, \infty) \subseteq B$.

2. Let $X$ be a metric space, of which $A$ and $B$ are compact subsets; define $\delta := \inf\{d(a,b) : a \in A, b \in B\}$. Prove that there exist $a_0 \in A$ and $b_0 \in B$ such that $d(a_0, b_0) = \delta$. Must $a_0$ and $b_0$ be unique? Proof or counterexample, as appropriate.

3. Let $f$ be a surjective map from the precompact metric space $X$ onto the metric space $Y$. Prove that if $f$ is *uniformly* continuous then $Y$ is also precompact. Show by example that if *uniformly* is dropped then $Y$ can fail to be precompact.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Can it be the case that for each $t \in \mathbb{R}$, the value $f(t)$ is irrational iff $t$ itself is rational? If yes, give an example; if no, give a proof.

5. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and assume that $f(t) \to \ell \in \mathbb{R}$ as $t \to \infty$. Prove that if $f'(t) \to L \in \mathbb{R}$ as $t \to \infty$ then $L = 0$.

---