First-year Analysis Examination
Part One
May 2017

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. Let $f : [0,1] \to [0,1]$ be increasing. By considering the set

$$A = \{t \in [0,1] : t \leqslant f(t)\}$$

or otherwise, prove that there exists $a \in [0,1]$ such that $f(a) = a$.

2. Let X be a metric space, of which U is an open subset and A an arbitrary subset. Prove that

$$U \cap \overline{A} \subseteq \overline{U \cap A}.$$

3. Let $f : X \to Y$ be a map between metric spaces. Prove that if $(f(x_n))_{n=0}^{\infty}$ converges in Y whenever $(x_n)_{n=0}^{\infty}$ converges in X then f is continuous.

[Note: it is **not** given that $f(x_n) \to f(x)$ whenever $x_n \to x$.]

4. Let $B \subseteq \mathbb{R}$ be bounded and $f : B \to \mathbb{R}$ uniformly continuous. True or false: $f(B) \subseteq \mathbb{R}$ is bounded? Give a proof or counterexample, as appropriate.

5. Let $f : (-a, a) \to \mathbb{R}$ be continuous everywhere and differentiable at each nonzero point. Show that if the limit $\ell = \lim_{t \to 0} f'(t)$ exists then f is in fact continuously differentiable at 0 with $f'(0) = \ell$.
