First-year Analysis Examination
Part One
January 2016

Answer FOUR questions in detail.
State carefully any results used without proof.

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1. For each λ in the nonempty index set Λ let X_λ be a nonempty subset of $\mathbb{R}$ and assume that the union $X = \bigcup\{X_\lambda : \lambda \in \Lambda\}$ is bounded above. Is $\sup\{\sup X_\lambda : \lambda \in \Lambda\}$ defined? Does it equal $\sup X$? Prove.

2. Let A be a subset of the metric space M and for $\delta > 0$ define

$$A_\delta = \{x \in M : (\exists a \in A)\ d(x, a) < \delta\}.$$

Prove (i) that A_δ is open and (ii) that $\cap_{\delta>0} A_\delta = \overline{A}$.

3. Let $K_0 \supseteq K_1 \supseteq \dots$ be a decreasing sequence of nonempty subsets of the metric space M.
(i) Prove that if each K_n is compact then $\cap_{n\geqslant 0} K_n$ is nonempty.
(ii) Show by example that if each K_n is closed then $\cap_{n\geqslant 0} K_n$ can be empty.

4. Let $f : X \to Y$ be a continuous bijection, with X complete. Prove that if f^{-1} is *uniformly* continuous then Y is complete. Give an example to show that dropping *uniformly* can jeopardize this conclusion.

5. For a differentiable function $f : (0, \infty) \to \mathbb{R}$ consider the implications:
(i) $\lim_{t\to\infty} f'(t) = 0 \Rightarrow \lim_{t\to\infty}(f(t)/t) = 0$;
(ii) $\lim_{t\to\infty}(f(t)/t) = 0 \Rightarrow \lim_{t\to\infty} f'(t) = 0$.
One is true and the other is false; prove the true and give a counterexample for the false.
