MAA 5229 First-Year Exam. May 2016

Do exactly four problems. Work should be presented in a logical fashion in order to receive full credit.

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1. Let $f(x) = \sum_{n=0}^{\infty} c_n x^n$ be convergent at $x = 2$ where the c_n are real. Prove that
$$f'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1}, \quad |x| < 2.$$

2. Let f be a real valued Riemannian integrable function on $[0,1]$ and let $F(x) = \int_0^x f(t)dt$. Prove $F'(x) = f(x)$ for almost all $x \in [0,1]$.

3. Let f be real and continuous on $[0,1]$. Prove, if $\int_0^1 f(x)x^n\,dx = 0$ for all $n = 0,1,2,...$, then $f \equiv 0$ on $[0,1]$.

4. Let f_n be a sequence of functions defined on $[0,1]$. Suppose there exists an $\alpha \in (0,1)$ and positive number M such that $|f_n(0)| \le M$ and $|f_n(x) - f_n(y)| \le |x-y|^\alpha$ for all n and $x,y \in [0,1]$. Prove there is a subsequence f_{n_k} of f_n converging (uniformly on $[0,1]$) to a function f that satisfies $|f(x) - f(y)| \le |x-y|^\alpha$ for all $x,y \in [0,1]$.

5. State Fatou's Theorem and give an example to show the statement can be false if the functions in the sequence are not assumed non-negative.