First-Year Analysis Examination, Part Two
January 2016

Do *exactly* two problems from Part A and two problems from part B. Answer each question on a separate sheet of paper. Write solutions in a neat and logical fashion, giving complete reasons for all steps.

Part A

1. Let $f_n : X \to \mathbb{R}$ be a uniformly convergent sequence of continuous functions on a compact metric space X. Prove that the set $\{f_n\}$ is equicontinuous on X.

2. Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series which converges for all $x \in \mathbb{R}$. State and prove the theorem concerning the uniform convergence of the series on finite intervals $[a, b]$. Must the series converge uniformly on all of $\mathbb{R}$? Prove, or give a counterexample.

3. Suppose $f \geq 0$, f is continuous on $[a, b]$, and $\int_a^b f(x)\,dx = 0$. Prove that $f(x) = 0$ for all $x \in [a, b]$.

Part B

1. Let $f_n : [0, 1] \to \mathbb{R}$ be a sequence of continuous functions. Prove that $g = \limsup f_n$ and $h = \liminf f_n$ are Lebesgue measurable.

2. For each of the following, either prove or give a counterexample (m denotes Lebesgue measure on $\mathbb{R}$; assume all functions are measurable):

   a) If f_n is integrable for all n, $f_n \to f$ uniformly on $\mathbb{R}$ and f is integrable, then $\int_{\mathbb{R}} f_n\,dm \to \int_{\mathbb{R}} f\,dm$.

   b) If $f_n \to f$ uniformly on $[0, 1]$ and f is integrable, then $\int_0^1 f_n\,dm \to \int_0^1 f\,dm$.

   c) Suppose $f_n \geq 0$ and $\int_0^1 f_n\,dm = 1$ for all n. If $f_n \to f$ pointwise, then $\int_0^1 f\,dm \leq 1$.

3. State the monotone convergence theorem and Fatou's theorem, and use the former to prove the latter.