MAA 5229 First-Year Exam, August 2015

Do exactly **2** problems from Part A and **2** problems from Part B. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

Part A

1. For $n \geq 1$ and $0 < t < 1$, let $f_n(t) = t^n$. Prove that:
   a) (f_n) converges uniformly on each compact subset of $(0, 1)$,
   b) (f_n) does not converge uniformly on $(0, 1)$.

2. Suppose $\mathcal{F}$ is an equicontinuous family of real-valued functions on $[0, 1]$, such that
$$\sup_{f \in \mathcal{F}} |f(0)| = M < +\infty.$$
   Prove that $\mathcal{F}$ is uniformly bounded (that is, that $\sup_{f \in \mathcal{F}} \sup_{x \in [0,1]} |f(x)| < +\infty$).

3. Prove that every monotone function $f : [a, b] \to \mathbb{R}$ is Riemann integrable.

Part B

1. State Fatou's lemma and use it to prove the Dominated Convergence Theorem.

2. Let (f_n) be a sequence of real-valued measurable functions on a measurable space $(X, \mathscr{M})$. Prove that each of the following subsets of X is measurable:
   a) $\{x \in X :$ the sequence $(f_n(x))$ is unbounded $\}$.
   b) $\{x \in X :$ the sequence $(f_n(x))$ is strictly increasing $\}$.

3. Let $g : [0, 1] \times [0, 1] \to \mathbb{R}$ be a function with the following properties:
   i) $|g(x, t)| \leq 1$ for all x and t,
   ii) for each t, the function $x \to g(x, t)$ is continuous on $[0, 1]$, and
   iii) for each x, the function $t \to g(x, t)$ is continuous on $[0, 1]$.

   Prove that the function h defined by
$$h(x) = \int_0^1 g(x, t)\, dt$$
   is continuous on $[0, 1]$.