MAA 5229 First-Year Exam, January 2015

Do exactly **2** problems from Part A and **2** problems from Part B. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

Part A

1. Let $\sum_{n=0}^{\infty} c_n x^n$ be a pwer series with finite radius of convergence R. Assume that the c_n and x are real valued. Let $\epsilon > 0$. State and prove the theorem concerning the uniform convergence of $\sum_{n=0}^{\infty} c_n x^n$ on $[-R+\epsilon, R-\epsilon]$.

2. Let $\{f_n\}$ be a pointwise bounded sequence of complex functions on a countable set E. Prove that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ such that $\{f_{n_k}(x)\}$ converges for each $x \in E$.

3. Let $\{f_n\}$ be a uniformly bounded sequence of Riemann integrable functions on $[a, b]$. Let
$$F_n(x) = \int_a^b f_n(t)\, dt, \quad a \le x \le b.$$
Prove that there exists a subsequence $\{F_{n_k}\}$ which converges uniformly on $[a, b]$.

Part B

1. Let E be a Lebesgue measurable subset of $[0, 1]$, with Lebesgue measure $m(E)$. If $0 \le \alpha \le m(E)$, show that there exists a Lebesgue measurable subset $F \subset E$ such that $m(F) = \alpha$. (*Hint:* Examine the function $f(x) = m(E \cap [0, x])$ on $[0, 1]$.)

2. Let $E \subset \mathbb{R}$ be Lebesgue measurable and suppose $m(E) < \infty$. It is known that for each $\epsilon > 0$ there is a closed set $F_\epsilon \subset E$ such that $m(E \setminus F_\epsilon) < \epsilon$. Show that F_ϵ can be chosen to be a compact set.

3. Let (X, Σ, μ) be a measure space and assume that f is μ-integrable. Prove that for $\epsilon > 0$ there exists a $\delta > 0$ such that $\left|\int_A f\, d\mu\right| < \epsilon$, whenever $A \in \Sigma$ and $\mu(A) < \delta$.