First-Year Examination
Analysis, Part Two
January 2014

Answer FOUR questions, starting each on a fresh sheet of paper. Write in a neat and logical fashion, giving complete reasons for all steps.

---

1. Let f be a continuous real-valued function on $[0,1]$. For each $n \geqslant 1$ let $h_n = n1_{[0,1/n]}$ be n times the indicator function of $[0,1/n]$. Prove that

$$\lim_{n\to\infty} \int_0^1 f(t)h_n(t)\mathrm{d}t = f(0).$$

2. Let f be a continuous real-valued function on $[0,1]$. For $n \geqslant 1$ and $0 \leqslant t \leqslant 1$ let $f_n(t) = f(t)t^n$. Determine a condition on $f(1)$ that is necessary and sufficient for the sequence $(f_n)_{n=1}^\infty$ to be uniformly convergent; be sure to prove necessity and sufficiency.

3. For $n \geqslant 1$ let f_n be a continuous real-valued function on $[0,1]$ and for $0 \leqslant t \leqslant 1$ define

$$F_n(t) = \int_0^t f_n(u)\mathrm{d}u.$$

In each of the following cases, decide whether $(F_n)_{n=1}^\infty$ must have a uniformly convergent subsequence, giving proof or counterexample as appropriate:
(i) $|f_n| \leqslant g$ for some continuous g and all n;
(ii) $0 \leqslant f_1 \leqslant f_2 \leqslant \dots$.

4. Let $(f_n)_{n=1}^\infty$ be a sequence of measurable functions on some measure space. Show that each of the following sets is measurable:

$$A = \{\omega | f_n(\omega) > 0 \text{ for infinitely many } n\};$$
$$B = \{\omega | f_n(\omega) \leqslant 0 \text{ for finitely many } n\}.$$

Warning: A and B need *not* be complementary!

5. Let $(f_n)_{n=1}^\infty$ be a uniformly-bounded sequence of continuous functions on $[0,1]$ that converges pointwise to zero. Does it follow that

$$\lim_{n\to\infty} \int_0^1 f_n(t)\mathrm{d}t = 0?$$

Does the answer change if ‘uniformly-bounded’ is removed from the hypotheses? Give proof or counterexample as appropriate.