MAA 5228 First-Year Exam. January 2016

Do exactly four problems. Work should be presented in a logical fashion in order to receive full credit.

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1. Let $A_1, ..., A_n, ...$ be subsets of a metric space X. If $B = \cup_{i=1}^n A_n$, prove that $\bar{B} = \cup_{i=1}^n \bar{A}_i$ where $\bar{B}$ stands for the closure of B. If $C = \cup_{n=1}^\infty A_n$, prove that $\cup_{i=1}^\infty \bar{A}_i \subset \bar{C}$.

2. Let f be a mapping from B_1 to itself, where B_1 is the unit ball in $\mathbb{R}^n$ ($n \geq 1$). Let $R(f)$ be the range of f in B_1. Prove that f is continuous if and only if for any open set V in B_1, $f^{-1}(V \cap R(f))$ is also open. Your reasoning should be based on the following definition of continuity: f is called to be continuous at x if for any given $\varepsilon > 0$, there exists $\delta > 0$ such that all $y \in B(x, \delta)$ (the ball centered at x with radius δ) satisfies $|f(y) - f(x)| < \varepsilon$.

3. Let K be a compact subset of $\mathbb{R}^n$ and F be a closed subset of $\mathbb{R}^n$ ($n \geq 1$). Suppose $K \cap F = \emptyset$, prove that

$$distance(K, F) := \inf_{x \in K, y \in F} |x - y| > 0.$$

   Moreover, there exist $x_0 \in K$ and $y_0 \in F$ such that $|x_0 - y_0| = distance(K, F)$.

4. Let f be a monotone function on $(0, 1)$. Prove that the number of points where f is discontinuous is at most countable.

5. Let f be continuous function from $[0, 2]$ to $[0, 2]$. Prove that f has a fixed point in $[0, 2]$. Give an example to show that if f is not continuous, the conclusion may not hold.

6. Suppose $\sum_{n=1}^\infty a_n$ is convergent and every $a_n \geq 0$ ($n = 1, .., \infty$), prove that $\sum_{n=1}^\infty \frac{\sqrt{a_n}}{n^{2/3}}$ is convergent. Give an example to show that $\sum_{n=1}^\infty \frac{a_n^{\frac{1}{4}}}{n^{\frac{2}{3}}}$ could be divergent.