# MAA 5228 First-Year Exam, August 2015

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Do exactly **4** problems. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

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1. Let $E$ be a dense ssubet of a metric space $X$ and let $U$ be an open subset of $X$. Prove that $U \subset \overline{E \cap U}$. (Here $\overline{E \cap U}$ denotes the closure of $E \cap U$.)

2. Let $X$ be a metric space and $K \subset X$ a compact subset. Prove that $K$ is closed.

3. Let $X$ be a compact metric space and $\mathcal{F}$ a collection of closed subsets of $X$ with the following property: *for all finite subcollections $\mathcal{G} \subset \mathcal{F}$, the intersection $\cap_{F \in \mathcal{G}} F$ is nonempty.* Prove that $\cap_{F \in \mathcal{F}} F$ is nonempty.

4. Let $X, Y$ be metric spaces and $f : X \to Y$ a continuous function. Prove that if $X$ is connected, then its image $f(X)$ is connected.

5. Let $(a_n)_{n \in \mathbb{N}}$ be a sequence of real numbers and $\sigma$ a permutation of $\mathbb{N}$. For each of the following provide either a proof or a counterexample.

   a) If the sequence $(a_n)_{n \in \mathbb{N}}$ converges then so does the rearrangement $(a_{\sigma(n)})_{n \in \mathbb{N}}$.

   b) If the series $\sum_{n \in \mathbb{N}} a_n$ converges then so does the rearrangement $\sum_{n \in \mathbb{N}} a_{\sigma(n)}$.

6. Let $f(x) = x^2 \sin \frac{1}{x}$ for $x \in \mathbb{R} \setminus \{0\}$ and $f(0) = 0$. Prove that $f$ is differentiable at every $x \in \mathbb{R}$. Is $f'$ continuous at $x = 0$? (Prove your claim.)