MAA 5228 First-Year Exam, January 2015

Do exactly **4** problems. Work must be presented in a neat and logical fashion in order to receive credit. Do not leave any gaps. When a theorem is used in a proof it must be precisely stated.

---

1. Let (X, d_X), (Y, d_Y) be metric spaces. On the Cartesian product $X \times Y := \{(x, y) : x \in X, y \in Y\}$ define the function

$$d_{X\times Y}((x_1, y_1), (x_2, y_2)) = d_X(x_1, x_2) + d_Y(y_1, y_2).$$

   It is known that $d_{X\times Y}$ is a metric on $X \times Y$. If X and Y are complete, must $X \times Y$ be complete? Prove or give a counterexample.

2. Let X be a metric space and suppose it has the following property: *whenever $\mathcal{C}$ is a collection of closed subsets of X, and $\cap_{C\in\mathcal{F}} C$ is nonempty for every finite $\mathcal{F} \subset \mathcal{C}$, then $\cap_{C\in\mathcal{C}} C$ is nonempty.* Prove that X is compact.

3. Let X be a metric space and A, B connected subsets of X. Prove that if $A \cap B$ is nonempty, then $A \cup B$ is connected.

4. a) Define what it means for a function $f : X \to Y$ to be uniformly continuous on a set $E \subset X$. b) Prove that if $f : X \to Y$ is continuous and X is compact, then f is uniformly continuous on X.

5. Let $f : X \to Y$ be a continuous bijection. Prove that if X is compact, then f^{-1} is continuous. Give an example to show that the conclusion can fail if X is not assumed compact.

6. Let $f : (a, b) \to \mathbb{R}$ be differentiable and suppose that f' is monotonically increasing. Prove that f' is continuous.