First-Year Examination
Analysis, Part One
January 2014

DO ONLY 4 PROBLEMS OF THE 6. PRINT NAME ON EACH SHEET. Work must be presented in a neat and logical fashion to receive credit. Do not leave any gaps. State clearly any theorems used in proofs.

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1. Let F be a closed subset of the reals $\mathbb{R}$ with the usual topology. Show F is a countable union of compact sets.

2. Let E be a nonempty subset of a metric space. State the definition of the distance $\rho_E(x)$ of a point x to E. Characterize $\overline{E}$ in terms of ρ_E, where $\overline{E}$ is the closure of E.

3. (a) Let f be a continuous function from a compact metric space X into a metric space Y. Prove $f(X)$ is compact.
(b) Assume the setting in (a) with the further assumption that Y is the reals with the usual topology. Show that f assumes maximum and minimum values on X.

4. Let f be a continuous map of a metric space X into a metric space Y.
(a) Show that if f is NOT uniformly continuous on X, then for some $\varepsilon > 0$ there are sequences $\{p_n\}, \{q_n\}$ in X such that $d_Y(f(p_n), f(q_n)) > \varepsilon$ for each n but $d_X(p_n, q_n) \to 0$.
(b) Assume that X is compact. Show, using (a), that f is uniformly continuous on X.

5. Let $\{x_n\}$ be a sequence of points in (a, b) and let $\{c_n\}$ be a sequence of positive numbers such that $\sum c_n$ converges. Define

$$f(x) = \sum_{n: x_n < x} c_n, \quad a < x < b,$$

where the summation is understood as follows: sum over those indices n for which $x_n < x$. If there are no points $x_n < x$, define the sum to be zero.
(a) Show $f(x-) = f(x)$ for each x in (a, b).
(b) Show $f(x_n+) - f(x_n-) = c_n$, for each n.

6. Suppose f is continuous on $[0, \infty)$, differentiable on $(0, \infty)$, $f(0) = 0$ and f' is monotonically increasing. Define g on $(0, \infty)$ by $g(x) = f(x)/x$, $x > 0$. Prove g is monotonically increasing.