First-Year Examination
Analysis, Part One
January 2013

Answer FOUR questions, starting each on a fresh sheet of paper. Write in a neat and logical fashion, giving complete reasons for all steps.

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1. Let $(a_n)_{n=1}^{\infty}$ be a real sequence and for $n \geqslant 1$ define

$$\alpha_n = (a_1 + \cdots + a_n)/n.$$

Prove that

$$\limsup_{n\to\infty} \alpha_n \leqslant \limsup_{n\to\infty} a_n$$

and show by example that the inequality can be strict.

2. Let X and Y be metric spaces; give $X \times Y$ the metric defined by

$$d((x,y),(a,b)) = d_X(x,a) + d_Y(y,b).$$

(i) If X and Y are compact, is $X \times Y$ compact? Proof or counterexample.
(ii) If X and Y are complete, is $X \times Y$ complete? Proof or counterexample.

3. Let $f : X \to X$ be a continuous map from a metric space to itself; assume that f has no fixed points. Prove that if X is compact then there exists $\varepsilon > 0$ such that $d(x, f(x)) \geqslant \varepsilon$ for each $x \in X$.

4. For a map $f : X \to Y$ between metric spaces, consider the statements:
(a) f is uniformly continuous;
(b) f maps Cauchy sequences to Cauchy sequences.
Of the implications $(a) \Rightarrow (b)$ and $(b) \Rightarrow (a)$, one is false and the other true; give a proof of the true and a counterexample for the false.

5. Let $f : (0, \infty) \to \mathbb{R}$ be differentiable and let b be a real number. Prove that if $f'(t) \to b$ as $t \to \infty$ then $f(t)/t \to b$ as $t \to \infty$.