Second Semester Algebra Exam **August 17, 2017**

Answer four problems, and on the list below circle the problems you wish to have graded. Do not circle more than four problems. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

Grade: 1 2 3 4 5 6

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1. Show that the ring $\mathbb{Z}[\sqrt{-5}]$ is not a principal ideal domain, and give an example of a non-principal ideal in this ring.

2. Let R be a commutative ring with identity. Show that an element $a \in R$ is contained in every maximal ideal if and only of $1 + ab$ is a unit in R for all $b \in R$.

3. Show that if p is an odd prime then the polynomial $\frac{(x+2)^p - 2^p}{x}$ is irreducible in $\mathbb{Q}[X]$.

4. Let K be a field. Show that any K-vector space V has a basis (do not assume that V has finite dimension).

5. Suppose R is a principal ideal domain and M is a finitely generated R-module. For any prime $p \in R$ let M_p be the set of $m \in M$ such that $p^n m = 0$ for some n. (i) Show that M_p is a submodule of M. (ii) Show that if $rM = 0$ for some nonzero $r \in R$ then M is isomorphic to the direct sum of the M_p for all primes p dividing r.

6. Find representatives of all conjugacy classes of elements of order 5 in $GL_2(\mathbb{F}_{19})$. Hint:
$$x^5 - 1 = (x-1)(x^2 - 4x + 1)(x^2 + 5x + 1)$$
in $\mathbb{F}_{19}[x]$, and the quadratic factors are irreducible.