First Semester Algebra Exam **August 17, 2017**

Answer four problems, and on the list below circle the problems you wish to have graded. Do not circle more than four problems. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

Grade: 1 2 3 4 5 6

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1. Let G be a finite p-group and suppose that G acts on a finite set S. Denote by S^G the subset of S fixed by every element of G. Show that $|S| \equiv |S^G| \pmod{p}$, where $|A|$ denotes the cardinality of the set A.

2. (i) (5 points) How many elements of S_6 have order 2? (ii) (5 points) How many elements of A_6 have order 2?

3. Let G be a finite group, and suppose d is an integer dividing the order of G. (i) (6 points) Show the number of elements of order d is divisible by $\phi(d)$ (Euler ϕ-function). (ii) (2 points) Give an example in which the number of elements of order d is zero. (iii) (2 points) Give an example in which the number of elements of order d is strictly larger than $\phi(d)$.

4. Show that a group of order $385 = 5 \cdot 7 \cdot 11$ has a normal cyclic subgroup of order 77.

5. Let G be a group that is the internal direct product of two *charactersitic* subgroups H and K. Show that $\mathrm{Aut}(G) \simeq \mathrm{Aut}(H) \times \mathrm{Aut}(K)$. Can the hypothesis that H and K are characteristic be omitted?

6. Let p be a prime and $n \geq 3$ an integer. Show that there exist nonabelian groups of order p^n with a cyclic subgroup of index p.