First Semester Algebra Exam **January 5, 2016**

Answer four problems, and on the list below circle the problems you wish to have graded. Write your answers clearly in complete English sentences. You may quote results (within reason) as long as you state them clearly.

Grade: 1 2 3 4 5 6 7

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1. Let G be a group. (i) (5 points) If $H \subseteq G$ is a normal subgroup and K is a characteristic subgroup of H, show that K is normal in G. (ii) (5 points) Give an example of a group and subgroups $K \subseteq H \subseteq G$ with H normal in G, K normal in H but not normal in G.

2. Let G be a group. (i) (5 points) Show that if G is abelian, the subset H of elements of finite order is a subgroup. (ii) (5 points) Give an example of a (nonabelian) group G and elements x, $y \in G$ such that xy has infinite order.

3. Show that a group of order $3^2 \cdot 17$ is abelian.

4. (i) (3 points) Define what it means for a group G to be *nilpotent*. (ii) (4 points) Show that any nilpotent group is solvable. (iii) (3 points) Give an example of a finite solvable group that is not nilpotent.

5. Let G be a group acting on a set S, and $H \subseteq G$ a normal subgroup. Assume that for any x_1, $x_2 \in S$ there is a *unique* $h \in H$ such that $h(x_1) = x_2$. For $x \in S$ let G_x be the stabilizer of x. Show that for any $x \in S$, G is a semi-direct product of G_x and H.

6. Show that for any $n \geq 5$, A_n is the only nontrivial proper normal subgroup of S_n.

7. Show that the normalizer of a 5-Sylow subgroup of A_5 has order 10, and is a maximal subgroup of A_5.